Skin-Effect Description in Electromagnetism with a Scaled Asymptotic Expansion

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Abstract

We study a transmission problem in high contrast media. The 3-D case of the Maxwell equations in harmonic regime is considered. We derive an asymptotic expansion with respect to a small parameter $\delta > 0$ related to high conductivity. This expansion is theoretically justified at any order. Numerical simulations highlight the skin-effect and the expansion accuracy.

Introduction

We consider the diffraction problem of waves by highly conducting materials in electromagnetism. The high conductivity reduces the penetration of the wave to a boundary layer, see [1]. The physical model is the following. Ω_{cd} is an open bounded domain in $\mathbb{R}^3$ with connected complement, occupied by a conducting medium. Ω_{cd} is embedded in an insulating medium Ω_{is}. We suppose that their common interface Σ is smooth. We define $\Omega = \Omega_{\mathsf{cd}} \cup \Sigma \cup \Omega_{\mathsf{is}}$. We denote by δ a small parameter which is inversely proportional to the square root of the conductivity σ. The depth of the boundary layer is proportional to δ. We first give the formal construction of the asymptotic expansion. Then, we prove optimal error estimates. Finally, we present numerical simulations in axisymmetric geometry.

1 Scaled asymptotic expansion

Eliminating the magnetic field $\mathbf{H}_\delta$ from Maxwell equations, we perform a study in electric field $\mathbf{E}_\delta$.

1.1 Normal coordinates

To describe the boundary layer in Ω_{cd}, we define a local normal coordinate system $(y_\alpha, y_3), \alpha \in \{1,2\}$, in a tubular neighborhood $\mathcal{O}$ of Σ, $y_3 \in (0, h_0)$. The euclidian metric in $\mathcal{O}$ is denoted by $(g_{ij})_{i,j\in\{1,2,3\}}$. We adopt the tensorial calculus, see [2], to write Maxwell equations in these coordinates. The curl operator writes

$$\left\{ \begin{array}{l} (\operatorname{curl} \mathbf{E})^\alpha = \frac{1}{\sqrt{\mathsf{g}}} \epsilon^{3\beta\alpha} (\partial_3 E_\beta - \partial_\beta E_3) \\ (\operatorname{curl} \mathbf{E})^3 = \frac{1}{\sqrt{\mathsf{g}}} \epsilon^{3\alpha\beta} D^h_\alpha E_\beta \end{array} \right.$$

with $\mathsf{g} = det(g_{ij})$, ϵ^{ijk} the Levi-Civita symbol, and D^h a covariant derivative defined for $h = y_3$, see [2].

1.2 Scaling and ansatz

We perform the scaling $Y_3 = \frac{y_3}{\delta}$ in $\mathcal{O}$, and expand the 3D Maxwell operator in power of series of δ. This leads to postulate the following expansions

$$\mathbf{E}^{\mathsf{is}}_\delta(\mathbf{x}) \sim \sum_{j\geq 0} \mathbf{E}^{\mathsf{is}}_j(\mathbf{x})\delta^j, \quad \mathbf{E}^{\mathsf{cd}}_\delta(\mathbf{x}) \sim \sum_{j\geq 0} \mathbf{E}^{\mathsf{cd}}_j(\mathbf{x};\delta)\delta^j \quad (1)$$

respectively in Ω_{is}, and Ω_{cd}, with $\mathbf{E}^{\mathsf{cd}}_j(\mathbf{x};\delta) = \mathbf{W}^{\mathsf{cd}}_j(y_\alpha, \frac{y_3}{\delta})$ when $(y_\alpha, y_3) \in \mathcal{O}$. We prove that $\mathbf{W}^{\mathsf{cd}}_0 = 0$, and $\mathbf{W}^{\mathsf{cd}}_j(., Y_3) = \exp(-\lambda Y_3)[a_0 + .. + a_{j-1}Y_3^{j-1}]$, with $\Re(\lambda) > 0$. According to Faraday's law $\mathbf{H}_\delta = 1/(i\kappa\mu_0) \operatorname{curl} \mathbf{E}_\delta$, $\kappa > 0$ (a wave number), we derive the expansion of the magnetic field.

2 Uniform a priori estimate

A regularized variational formulation of our problem in the domain Ω using the space $\mathbf{X_N}(\delta) = \{\mathbf{E} \in \mathbf{H}(\operatorname{curl},\Omega) | \ \ \varepsilon(\delta)\mathbf{E} \in \mathbf{H}(\operatorname{div},\Omega), \ \ \mathbf{E}\times\mathbf{n} = 0 \ \ \text{on} \ \ \partial\Omega\}$ is:
Find $\mathbf{E}^\delta \in \mathbf{X_N}(\delta)$ such that for all $\mathbf{u} \in \mathbf{X_N}(\delta)$

$$\int_\Omega \operatorname{curl} \mathbf{E} \cdot \operatorname{curl} \overline{\mathbf{u}} + \operatorname{div} \varepsilon(\delta)\mathbf{E} \operatorname{div} \overline{\varepsilon(\delta)\mathbf{u}} - \kappa^2 \varepsilon(\delta)\mathbf{E}\cdot\overline{\mathbf{u}} \mathrm{d}\mathbf{x} = \int_\Omega (\mathbf{F}^\delta \cdot \overline{\mathbf{u}} - \frac{1}{\kappa^2} \operatorname{div} \mathbf{F}^\delta \operatorname{div} \overline{\varepsilon(\delta)\mathbf{u}}) \mathrm{d}\mathbf{x}, \quad (2)$$

with $\varepsilon(\delta) = \frac{1}{\mu_0}(\mathbf{1}_{\Omega_{\mathsf{is}}} + (1 + \frac{i}{\delta^2})\mathbf{1}_{\Omega_{\mathsf{cd}}})$, $\mu_0 > 0$, and $\mathbf{F}^\delta \in \mathbf{H}(\operatorname{div},\Omega)$. Under a spectral hypothesis on κ, we prove that $\exists\, \delta_0 > 0$ small enough such that $\forall \delta \in (0, \delta_0)$, (2) has a unique solution $\mathbf{E}^\delta \in \mathbf{X_N}(\delta)$, and there is a constant $C > 0$ independent of δ such that

$$\| \operatorname{curl} \mathbf{E}^\delta \|_{0,\Omega} + \| \operatorname{div} \varepsilon(\delta) \mathbf{E}^\delta \|_{0,\Omega} + \| \mathbf{E}^\delta \|_{0,\Omega} + \frac{1}{\delta} \| \mathbf{E}^\delta \|_{0,\Omega_{\mathsf{cd}}} \leq C \| \mathbf{F}^\delta \|_{\mathbf{H}(\operatorname{div},\Omega)}.$$

We prove this estimate using a technique of vector potential, see [3]. Defining $\mathbf{R}^\delta_m$ from (1) by removing to $\mathbf{E}_\delta$ the $m+1$ first-terms of the expansion, we deduce optimal error estimates for the truncated expansions

$$\| \mathbf{R}^\delta_m \|_{\mathbf{H}(\operatorname{curl},\Omega)} + \| \operatorname{div} \varepsilon(\delta) \mathbf{R}^\delta_m \|_{0,\Omega} + \frac{1}{\delta} \| \mathbf{R}^\delta_m \|_{0,\Omega_{\mathsf{cd}}} \leq C\delta^{m-1}.$$

3 Numerical simulations

Here we present numerical experiments to illustrate the accuracy of the asymptotic expansion. We perform our analysis to the magnetic field and restrict ourselves to axisymmetric domains with an orthoradial data. After having meshed the domain, we compute the orthoradial component of the magnetic field H_θ with the finite element library Melina, see [4]. The numerical method uses a variational form with unknown $\mathsf{H}_\theta(r,z)$ in a meridian domain, which is meshed in such a way the boundary layer is correctly taken into account, see Figure 1 for spheroidal domains. We extract values of $|H_\theta(y_3)|$

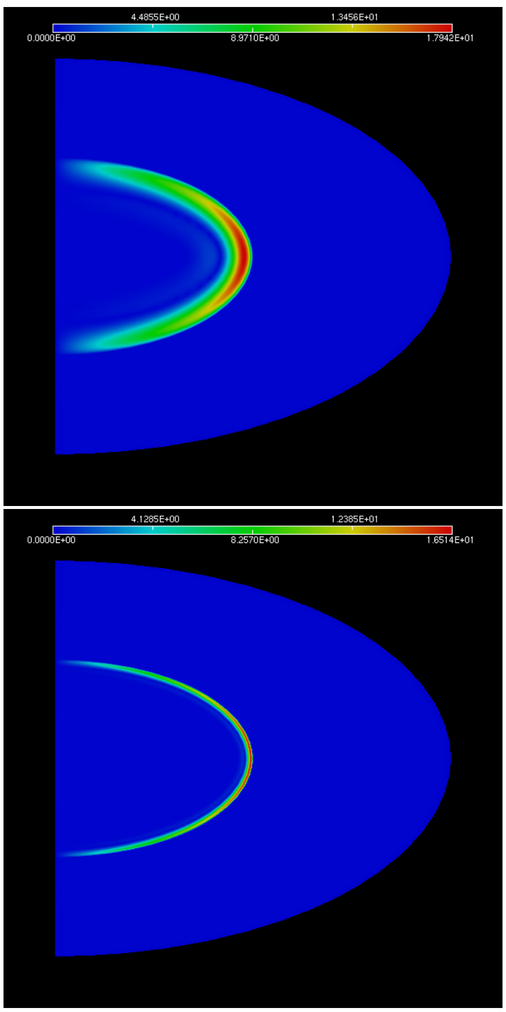

Figure 1: $|\Im H_\theta|$ for $\sigma = 5\,S.m^{-1}$ and $80\,S.m^{-1}$

along edges of the mesh for $z = 0$. We perform a linear regression from values of $\log_{10}|H_\theta(y_3)|$ in a depth-skin defined by $skin(\sigma) = \sqrt{2}\delta(\sigma)/\kappa$, corresponding to $n(\sigma)$ points on the mesh, see Figure 2 for spherical domains with $y_3 = 2 - r$. We

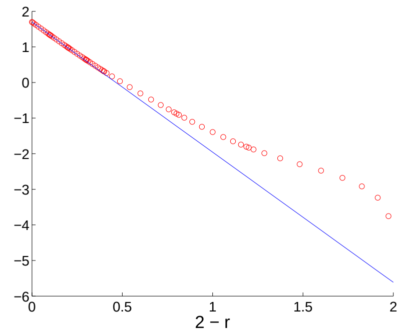

Figure 2: $\log_{10}|H_\theta(2-r)|$ for $\sigma = 5\,S.m^{-1}$

get a "numerical" slope $s(\sigma)$. Let $\mathsf{H}^{\mathsf{cd}}_{\theta,1}$ be the orthoradial component of $\mathbf{H}^{\mathsf{cd}}_0(\mathbf{x};\delta) + \delta\mathbf{H}^{\mathsf{cd}}_1(\mathbf{x};\delta)$ (the truncated asymptotic expansion of $\mathbf{H}_\delta$). We prove that $\log_{10}|H^{\mathsf{cd}}_{\theta,1}(y_3)| = \alpha(\sigma)y_3 + \log_{10}|H^{\mathsf{is}}_{\theta,0|\Sigma}| + \mathcal{O}(\delta)$. $\alpha(\sigma) = \big(\beta - 1/skin(\sigma)\big)/\ln 10$ is a "theoretical" slope, and β depends on the curvature on Σ. The accuracy of the expansion is tested by representing the relative error between "numerical" and "theoretical" slopes: $error(\sigma) = |\alpha(\sigma) - s(\sigma)|/|\alpha(\sigma)|$, see Figure 3.

σ $(S.m^{-1})$	5	20	80
$skin(\sigma)$ (cm)	10.3	5.15	2.58
$n(\sigma)$	7	5	3
$s(\sigma)$	$-3,65542$	$-7,88390$	$-16,27916$
$\alpha(\sigma)$	$-3,67331$	$-7,88950$	$-16,32188$
$error(\sigma)$ (%)	0, 48	0, 07	0, 26

Figure 3: Relative error in spheroidal domains

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